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A VARIATIONAL SOLUTION TO THE TAYLOR STABILITY PROBLEM BASED UPON NON-EQUILIBRIUM THERMODYNAMICS

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Abstract—The recent papers of Glansdorff, Prigogine and Schechter have shown that a time dependent local potential exists in which the macroscopic time evolution corresponds to an extremum. Schechter and Himmelblau [19] solved the well-known Bénard problem using a restricted variational formulation. In this paper the works of Glansdorff, Prigogine and Schechter are modified and extended to obtain a variational solution to the Taylor stability problem for a viscous fluid contained between rotating coaxial cylinders with and without a radial temperature gradient. The results obtained are in excellent agreement with the analytical results of Chandrasekhar [9], Chandrasekhar and Elbert [10], Diprima, Walowit and Tsao [4], Yih [5], Lai [25], and Becker and Kaye [7] and with the experimental results of Becker and Kaye [8], Bjorklund and Kays [12], Haas and Nissan [13] and Ho, Nardacci and Nissan [15].

NOMENCLATURE

a,	dimensionless wave number;
A,	constant equation (24);
A_1,	constant equation (49);
B,	constant equation (25);
B_1,	constant equation (49);
$C^{(k)}_{(\alpha)}$,	constants equation (43);
c_v,	specific heat at constant volume;
e,	internal energy per unit mass;
J_i,	ith thermodynamic flux;
k,	thermal conductivity;
L,	functional symbol;
l,	$R_2 - R_1$;
N,	ratio of Rayleigh to Taylor number;
Pr,	Prandtl number;
p,	pressure;
Ra,	Rayleigh number;
R_1, R_2,	radius of inner and outer cylinder;
r,	radius;
S,	surface;
Ta,	Taylor number;
T,	temperature perturbation;
T_1, T_2,	temperature of inner and outer cylinder;
t,	time;
u_i,	ith component of velocity vector;
u,	radial perturbation component of velocity;
V,	volume;
v,	θ perturbation component of velocity;
W_i,	ith component of heat flux vector;
x,	dimensionless radius, equation (47);
x_i,	ith cartesian coordinate;
X_i,	ith component of body force;
$X_{(\alpha)}$,	generalized perturbation amplitude;
z,	axial coordinate.

Greek symbols

α,	coefficient of volume expansion;
α_n,	constant in equation (57);
β_n,	constant in equation (58);
γ_n,	constant in equation (59);

δ,	variational symbol;
δ_{ij},	Kronecker delta;
ε,	time interval;
ζ,	dimensionless length;
η,	radius ratio R_1/R_2;
$\tilde{\eta}$,	pressure perturbation amplitude;
θ,	cylindrical coordinate;
$\tilde{\theta}$,	temperature perturbation amplitude;
λ,	wavelength in axial direction z;
μ,	absolute viscosity;
$\tilde{\mu}$,	speed ratio Ω_2/Ω_1;
ν,	kinematic viscosity μ/ρ;
ξ_x, ξ_θ, ξ_z,	velocity perturbation amplitudes;
ρ,	density;
$\tilde{\sigma}$,	stability parameter;
τ_{ij},	ith, jth component of stress tensor;
ϕ,	local potential;
Ω_1, Ω_2,	angular velocity of inner and outer cylinder.

Superscripts

*,	nonvaried quantity during variational process;
s,	evaluated at a stationary state.

Subscripts

0,	evaluation at a reference state;
i,	ith component of a vector;
ij,	ith, jth component of a tensor;
r, x, θ, z,	components in coordinate directions.

I. INTRODUCTION

THE PROBLEM of the stability of a viscous fluid contained between concentric cylinders was first analyzed by G. I. Taylor [1] in 1923. Since then numerous researchers have obtained solutions under a wide variety of boundary conditions and restraints. An excellent review of the research connected with this problem has been compiled by Chandrasekhar [2] and by DiPrima [3]. In 1964, Diprima, Walowit and Tsao [4] solved the Taylor problem using Galerkin's Method with and without a radial temperature gradient and obtained results which were in good agreement with the analytical results of Yih [5], Lai [6], Becker and Kaye [7, 8], Chandrasekhar [9], Chandrasekhar and Elbert [10], and Kirchgassner [11]. In addition their results were in good agreement with the experimental results of Becker and Kaye [8], Bjorklund and Kays [12], Haas and Nissan [13], and Ho, Nardacci and Nissan [14].

Because of the excellent agreement between the published analytical and experimental results, this problem was selected to demonstrate the usefulness of a variational technique which is based upon non-equilibrium thermodynamics. In 1964 Prigogine and Glansdorff [15] showed that for the whole class of macroscopic systems subjected to time independent boundary conditions, there existed a general criterion of evolution. In this paper, the evolution criterion predicted the existence of a quantity $\mathrm{d}\phi$ of the form

$$\mathrm{d}\phi = \int_V \sum_i J_i' \,\mathrm{d}X_i' \mathrm{d}V \leqslant 0 \tag{1}$$

where the forces X_i' and fluxes J_i' include mechanical processes (i.e. convection terms) as well as dissipative processes normally associated with entropy production. In the case where only dissipation processes occur, equation (1) implies the theorem of minimum entropy production which was previously proposed by Prigogine [16]. When $\mathrm{d}\phi$ becomes a total differential, equation (1) is referred to as the local potential. The practical importance of the local potential arises from the possibility of determining the stationary states through a variational principle. Numerous examples of variational solutions to boundary value problems can be found in the works of Hays [17, 18], Schechter [19] and Butler and Rackley [20].

In 1965 Glansdorff and Prigogine [21] showed that the concept of the local potential was closely related to fluctuation theory. In this paper they defined time-dependent local potentials which were such that the macroscopic time evolution corresponded to an extremum. Schechter and Himmeblau [19] found a local potential for the Benard Problem and obtained a variational solution.

In the present work a local potential for the Taylor problem is obtained using the techniques proposed by Prigogine and Glansdorff [21]. A direct method is then applied to the generalized variational integral to obtain a solution to the Taylor problem with and without an imposed radial temperature gradient.

II. GENERALIZED LOCAL POTENTIAL

Using the techniques proposed by Glansdorff and Prigogine [21] a functional form which includes variable conductivity and viscosity can be found. Consider an incompressible fluid in a volume which has a boundary surface S. The equations of continuity momentum and energy when written in cartesian tensor form are:

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho X_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij}) \tag{3}$$

$$\rho\left(\frac{\partial e}{\partial t} + u_j\frac{\partial e}{\partial x_j}\right) = -\frac{\partial W_i}{\partial x_i} + \tau_{ij}\frac{\partial u_i}{\partial x_j} - p\frac{\partial u_j}{\partial x_j} \tag{4}$$

where

$$W_i = -k\frac{\partial T}{\partial x_i} \tag{5}$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{6}$$

If equations (2), (3) and (4) are multiplied by infinitesimal variations δp, $-\delta u_i$, and $(1/T_0)\,\delta T$ respectively and then added, the resulting equation can be integrated by parts and then integrated over the volume V and time t to yield

$$-\int_t\int_v\left(\rho\frac{\partial u_i}{\partial t}\delta u_i + \frac{\rho c_v}{T_0}\frac{\partial T}{\partial t}\delta T\right)\mathrm{d}V\,\mathrm{d}t$$
$$= \int_t\int_v\left[\frac{\partial u_i}{\partial x_i}\delta p + \frac{\partial p}{\partial x_i}\delta u_i + \frac{p}{T_0}\frac{\partial u_i}{\partial x_i}\delta T\right.$$
$$+\frac{\mu}{2}\delta\left(\frac{\partial u_i}{\partial x_j}\right)^2 - \frac{\partial\mu}{\partial x_j}\frac{\partial u_j}{\partial x_i}\delta u_i + \frac{k}{2T_0}\delta\left(\frac{\partial T}{\partial x_i}\right)^2$$
$$+\frac{u_i}{T_0}\left(\frac{\partial\mu}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)\delta T$$
$$+\frac{\mu u_i}{T_0}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta\frac{\partial T}{\partial x_j} + \frac{\rho u_j}{2}\delta\frac{\partial}{\partial x_j}u_iu_i$$
$$\left.-\rho u_ju_i\,\delta\frac{\partial u_i}{\delta x_j} + \frac{\rho c_v u_j}{T_0}\frac{\partial T}{\partial x_j}\delta T - \rho X_i\delta u_i\right]$$
$$\times\,\mathrm{d}V\,\mathrm{d}t - \int_t\int_s\left[\mu\frac{\partial u_i}{\partial x_j}\delta u_i + \frac{k}{T_0}\frac{\partial T}{\partial x_j}\delta T\right.$$
$$\left.+\frac{\mu}{T_0}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)u_i\,\delta T\right]\eta_j\,\mathrm{d}S\,\mathrm{d}t. \tag{7}$$

When the families of temperature and velocity distributions consist of an appropriate macroscopic distribution plus small and arbitrary variations around the macroscopic distribution such that

$$T(x_i, t) = T^*(x_i, t) + \delta T \tag{8}$$

$$u_i(x_i, t) = u_i^*(x_i, t) + \delta u_i \tag{9}$$

and when the thermal conductivity and viscosity are functions of T and can be represented by

$$k(T) = k^*(T^*) + \delta k \tag{10}$$

$$\mu(T) = \mu^*(T^*) + \delta\mu \tag{11}$$

then equation (7) becomes:

$$\delta L = -\frac{1}{2}\int_v\left[\rho(\delta u_i)^2 + \frac{\rho c_v}{T_0}(\delta T)^2\right]\mathrm{d}V \leqslant 0$$

$$\delta L = \int_t\int_v\left\{\rho\frac{\partial u_i^*}{\partial t}\delta u_i + \frac{\rho c_v}{T_0}\frac{\partial T^*}{\partial t}\delta T + \frac{\partial u_i^*}{\partial x_i}\delta p\right.$$
$$+\frac{\partial p^*}{\partial x_i}\delta u_i + \frac{p^*}{T_0}\frac{\partial u_i^*}{\partial x_i}\delta T + \frac{\mu^*}{2}\delta\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)$$
$$-\frac{\partial\mu^*}{\partial x_j}\frac{\partial u_j^*}{\partial x_i}\delta u_i + \frac{k^*}{2T_0}\delta\left(\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_i}\right)$$
$$+\frac{u_i^*}{T_0}\left[\frac{\partial\mu^*}{\partial x_j}\frac{\partial u_i^*}{\partial x_j} + \mu^*\frac{\partial}{\partial x_j}\left(\frac{\partial u_i^*}{\partial x_j}\right)\right]\delta T$$
$$+\frac{\mu^* u_i^*}{T_0}\left[\frac{\partial u_i^*}{\partial x_i} + \frac{\partial u_j^*}{\partial x_j}\right]\delta\frac{\partial T}{\partial x_j} + \frac{\rho u_j^*}{2}\delta\frac{\partial}{\partial x_j}u_iu_i$$

$$- \rho u_j^* u_i^* \delta \frac{\partial u_i}{\partial x_j} + \frac{\rho c_v}{T_0} u_j^* \frac{\partial T^*}{\partial x_j} \delta T - \rho X_i^* \delta u_i \Big\} \mathrm{d}V \,\mathrm{d}t - \iint_{t\, s} \Big[\mu^* \frac{\partial u_i^*}{\partial x_j} \delta u_i + \left\{ \frac{k^*}{T_0} \frac{\partial T^*}{\partial x_j} + \frac{\mu^* u_i^*}{T_0} \left(\frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i} \right) \right\} \delta T \Big] n_j \,\mathrm{d}S \,\mathrm{d}t \leqslant 0 \quad (12)$$

As a result of equation (12), a generalized functional for an incompressible fluid with variable conductivity and viscosity can be written as:

$$L = \iint_{t\, v} \Big[\rho \frac{\partial u_i^*}{\partial t} u_i + \frac{\rho c_v}{T_0} \frac{\partial T^*}{\partial t} T + \frac{\partial u_i^*}{\partial x_i} p + \frac{\partial p^*}{\partial x_i} u_i + \frac{p^*}{T_0} \frac{\partial u_i^*}{\partial x_i} T + \frac{\mu^*}{2} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right) - \frac{\partial \mu^*}{\partial x_j} \frac{\partial u_j^*}{\partial x_i} u_i + \frac{k^*}{2T_0} \left(\frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_i} \right) + \frac{u_i^*}{T_0} \left(\frac{\partial \mu^*}{\partial x_j} \frac{\partial u_i^*}{\partial x_j} + \mu^* \frac{\partial}{\partial x_j} \frac{\partial u_i^*}{\partial x_j} \right) T + \frac{\mu^* u_i^*}{T_0} \left(\frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i} \right) \frac{\partial T}{\partial x_j} + \frac{\rho u_j^*}{2} \frac{\partial}{\partial x_j} u_i u_i - \rho u_j^* u_i^* \frac{\partial u_i}{\partial x_j} + \frac{\rho c_v u_j^*}{T_0} \frac{\partial T^*}{\partial x_j} T - \rho X_i^* u_i \Big] \mathrm{d}V \,\mathrm{d}t - \iint_{t\, s} \Big[\mu^* \frac{\partial u_i^*}{\partial x_j} u_i + \left\{ \frac{k^*}{T_0} \frac{\partial T^*}{\partial x_i} + \frac{\mu^* u_i^*}{T_0} \times \left(\frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i} \right) \right\} T \Big] n_j \,\mathrm{d}s \,\mathrm{d}t \quad (13)$$

In this form the functional contains the time-dependent quantities of interest (u_i^*, T^*, p^*) and fluctuating quantities u_i, T and p. In a variational sense we are then dealing with a restricted functional which involves two types of terms; those which may vary during the evolution of the system and those which remain constant during the variations process. As a necessary condition, the Euler–Lagrange equations

$$\left.\frac{\delta L}{\delta T}\right)_{u_i, u_i^*, T^*, p, p^*} = 0 \quad (14)$$

$$\left.\frac{\delta L}{\delta p}\right)_{u_i, u_i^*, T, T^*, p^*} = 0 \quad (15)$$

$$\left.\frac{\delta L}{\delta u_i}\right)_{u_i^*, T, T^*, p, p^*} = 0 \quad (16)$$

yield the continuity, momentum and energy equations after the subsidiary conditions

$$\begin{aligned} u_i &= u_i^* \\ T &= T^* \\ p &= p^* \end{aligned} \quad (17)$$

are satisfied. Since the starred and unstarred quantities are functions of position and time, the techniques used for the solution of such a variational problem will differ from those normally employed in classical variational problems.

III. THE VARIATIONAL FORM OF THE TAYLOR PROBLEM

To obtain the functional for the Taylor problem the following assumptions are made: (1) The thermal conductivity, specific heat and viscosity of the fluid are assumed constant. As a result the analysis is limited to gases with small temperature differences or liquids over slightly larger differences; (2) The viscous dissipation and energy associated with the change in pressure have been neglected; (3) The gravitational field effects are assumed negligible in comparison to the centrifugal force field effects; (4) The perturbations are arbitrary infinitesimal disturbances and occur over a very short interval of time; (5) The perturbations are assumed to be axisymmetric; and (6) The perturbations in the axial direction are periodic to account for the infinite boundary condition in the axial direction. Under these restrictions, the functional (13) becomes:

$$L = \iint_{t\, v} \Big[\rho \frac{\partial u_i^*}{\partial t} u_i + \frac{\rho c_v}{T_0} \frac{\partial T^*}{\partial t} T + \frac{\partial u_i^*}{\partial x_i} p$$

$$+ \frac{\partial p^*}{\partial x_i} u_i + \frac{p^*}{T_0}\frac{\partial u_i^*}{\partial x_i} T + \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)$$
$$+ \frac{k}{2T_0}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_i} + \frac{\rho u_j^*}{2}\frac{\partial}{\partial x_j} u_i u_i$$
$$\left. - \rho u_j^* u_i^* \frac{\partial u_i}{\partial x_j} + \frac{\rho c_v u_j^*}{T_0}\frac{\partial T^*}{\partial x_j} T\right] \mathrm{d}V\,\mathrm{d}t$$
$$- \iint\limits_{t\ s} \left[\mu \frac{\partial u_i^*}{\partial x_j} u_i + \frac{k}{T_0}\frac{\partial T^*}{\partial x_j} T\right] n_j\,\mathrm{d}s\,\mathrm{d}t. \qquad (18)$$

Due to the boundary conditions on the varying quantities u_i and T and because of the periodic nature of the z component, the surface integral may also be neglected in (18). For a small interval of time ε it is assumed [19, 22] that the functional L in cylindrical coordinates (r, z) can be approximated to the second order in ε by the Taylor series expansion:

$$L \cong \varepsilon I_1 + \varepsilon^2/2\, I_2$$
$$I_1 = \left[\!\!\left[\iint\limits_{r\ z} \left\{ \rho \frac{\partial u_r^*}{\partial t} u_r + \rho \frac{\partial u_\theta^*}{\partial t} u_\theta + \rho \frac{\partial u_z^*}{\partial t} u_z \right.\right.\right.$$
$$+ \frac{\rho c_v}{T_0}\frac{\partial T^*}{\partial t} T + \left(\frac{\partial u_r^*}{\partial r} + \frac{u_r^*}{r} + \frac{\partial u_z^*}{\partial z}\right) p + \frac{\partial p^*}{\partial r} u_r$$
$$+ \frac{\partial p^*}{\partial z} u_z + \rho\left(u_r^* \frac{\partial u_r^*}{\partial r} + u_z^* \frac{\partial u_r^*}{\partial z}\right) u_r - \rho \frac{u_\theta^{*2}}{r} u_r$$
$$+ \rho\left(u_r^* \frac{\partial u_\theta^*}{\partial r} + u_z^* \frac{\partial u_\theta^*}{\partial z} + \frac{u_r^* u_\theta^*}{r}\right) u_\theta$$
$$+ \rho\left(u_r^* \frac{\partial u_z^*}{\partial r} + u_z^* \frac{\partial u_z^*}{\partial z}\right) u_z + \frac{\nu\rho}{2}\left[\left(\frac{\partial u_r}{\partial r}\right)^2\right.$$
$$+ \left(\frac{\partial u_\theta}{\partial r}\right)^2 + \left(\frac{\partial u_2}{\partial r}\right)^2 + \left(\frac{u_r}{r}\right)^2 + \left(\frac{u_\theta}{r}\right)^2$$
$$\left. + \left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] + \frac{\rho c_v}{T_0}$$
$$\times \left(u_r^* \frac{\partial T^*}{\partial r} T + u_z^* \frac{\partial T^*}{\partial z} T\right) + \frac{k}{2T_0}$$
$$\times \left.\left.\left.\left[\left(\frac{\partial T}{\partial r}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right]\right\} r\,\mathrm{d}r\,\mathrm{d}z\right]\!\!\right]_{t=0}$$
$$I_2 = \left[\frac{\partial}{\partial t}\iint\limits_{r\ 2} [\text{same integrand as } I_1]\, r\,\mathrm{d}r\,\mathrm{d}z\right]_{t=0} \qquad (19)$$

In the case of rotating coaxial cylinders where u_r, u_θ, u_z denote the components of the velocity in the increasing r, θ and z directions, the Navier-Stokes and energy equations admit stationary solutions of the form:

$$u_r^{(s)} = u_z^{(s)} = 0 \qquad (20)$$
$$u_\theta^{(s)} = V^{(s)} = Ar + \frac{B}{r} \qquad (21)$$
$$T^{(s)} = T_1 - \frac{(T_2 - T_1)}{\ln \eta}\ln\frac{r}{R_1} \qquad (22)$$
$$\frac{\partial p^{(s)}}{\partial r} = \rho\frac{(V^{(s)})^2}{r} \qquad (23)$$

where

$$A = \frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_2^2 - R_1^2} \qquad (24)$$
$$B = R_1^2 R_2^2\left(\frac{\Omega_1 - \Omega_2}{R_2^2 - R_1^2}\right) \qquad (25)$$
$$\eta = \frac{R_1}{R_2} \qquad (26)$$

and

$$\tilde{\mu} = \frac{\Omega_2}{\Omega_1} \qquad (27)$$

Now let the stationary state be perturbed by infinitesimal disturbances such that

$$u_r = \xi_r \exp(-\tilde{\sigma}t) \qquad (28)$$
$$u_\theta = V^{(s)} + \xi_\theta \exp(-\tilde{\sigma}t) \qquad (29)$$
$$u_z = \xi_z \exp(-\tilde{\sigma}t) \qquad (30)$$
$$T = T^{(s)} + \tilde{\theta}\exp(-\tilde{\sigma}t) \qquad (31)$$
$$p = p^{(s)} + \tilde{\eta}\exp\ (-\tilde{\sigma}t) \qquad (32)$$

and

$$u_r^* = \xi_r\ \exp(-\tilde{\sigma}^* t) \qquad (33)$$
$$u_\theta^* = V^{(s)} + \xi_\theta^* \exp(-\tilde{\sigma}^* t) \qquad (34)$$
$$u_z^* = \xi_z^* \exp(-\tilde{\sigma}^* t) \qquad (35)$$
$$T^* = T^{(s)} + \tilde{\theta}^* \exp(-\tilde{\sigma}^* t) \qquad (36)$$

$$p^* = p^{(s)} + \tilde{\eta}^* \exp(-\tilde{\sigma}^* t) \tag{37}$$

where the density variation due to temperature change is given by

$$\rho = \rho_0 [1 + \alpha(T_0 - T)] \tag{38}$$

where α is the coefficient for volume expansion.

For small temperature variations, the variation in density will be small and the density will be assumed constant except when multiplied by the centrifugal acceleration term $(v^s)^2/r$. Here the density disturbance will be given by the well-known Boussinesq approximation

$$\delta\rho = -\rho_0 \alpha \tilde{\theta}^* \exp(-\tilde{\sigma}^* t). \tag{39}$$

It is to be noted that when the assumed perturbations [equations (28–38)] are inserted into equation (19), the function L is not extremalized since the perturbations ξ_r, ξ_θ, ξ_z, $\tilde{\eta}$ and $\tilde{\theta}$ as well as the stability parameter $\tilde{\sigma}$ are as yet undefined. Thus these functions and the stability parameter are selected such that L is extremalized. A necessary condition for this extremalization is that the first variation vanish.

$$\delta L = \varepsilon \sum_{\alpha=1}^{n} \frac{\delta I_1}{\delta X_{(\alpha)}} \delta X_{(\alpha)} + \frac{\varepsilon^2}{2} \sum_{k=1}^{n} \frac{\delta I_2}{\delta X_{(\alpha)}} \delta X_{(\alpha)} + \frac{\varepsilon^2}{2} \frac{\partial I^2}{\partial \tilde{\sigma}} \mathrm{d}\tilde{\sigma} \tag{40}$$

where the $X_{(\alpha)}$ are the perturbations of the dependent variables. Thus, for $\delta L = 0$ it is necessary for

$$\frac{\delta I_1}{\delta X_{(\alpha)}} = 0; \qquad \frac{\delta I_2}{\delta X_{(\alpha)}} = 0 \tag{41}$$

and

$$\frac{\delta I_2}{\delta \tilde{\sigma}} = 0 \tag{42}$$

In general the sign of $\tilde{\sigma}$ can not be determined without specific knowledge about the form of the perturbations $X_{(\alpha)}$. Here a Ritz [22] type of approximation is assumed such that

$$X_{(\alpha)} = \sum_{k=1}^{r} C_\alpha^k \phi_k \tag{43}$$

where the C_α^k are constants and the ϕ_k are functions of the spatial coordinates and members of a complete set which satisfy the boundary conditions for all values of the arbitrary constants C_α^k. A similar set of functions is assumed for the unvaried perturbations.

The coefficients C_α^k can be found from the system of equations

$$\frac{\partial I_1}{\partial C_\alpha^k} = 0 \qquad \begin{matrix} \alpha = 1, 2, \ldots, n \\ k = 1, 2, \ldots, r. \end{matrix} \tag{44}$$

After the differentiation the subsidiary conditions

$$X_{(\alpha)} = X_{(\alpha)}^* \tag{45}$$

$$\tilde{\sigma} = \tilde{\sigma}^* \tag{46}$$

are satisfied and the resulting $(n \times r)$ equations yield an eigenvalue problem having $\tilde{\sigma}$ as a parameter. The state of marginal stability can be found by allowing $\tilde{\sigma}$ to vanish.

To solve the Taylor problem, let us introduce a pair of dimensionless quantities such that

$$x = \frac{r - R_0}{l} \tag{47}$$

and

$$a = \lambda l \tag{48}$$

where

$$l = R_2 - R_1$$

λ = wavenumber in the axial direction

$$R_0 = \frac{(R_1 + R_2)}{2}$$

Thus the angular velocity becomes

$$\Omega = \Omega_1 g(x) = \Omega_1 \left[A_1 + \frac{B_1}{\zeta^2}\right] \tag{49}$$

where

$$\zeta = \frac{r}{R_2} = \eta + (1 - \eta)(x + \tfrac{1}{2})$$

$$A_1 = \frac{(\tilde{\mu} - \eta^2)}{(1 - \eta^2)}$$

$$B_1 = \frac{\eta^2(1 - \tilde{\mu})}{1 - \eta^2}$$

When the perturbations [equations (28–38)] and the dimensionless radius are introduced into I_1, it becomes:

$$I_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^l \Bigg[-\rho\tilde{\sigma}^* \left(\xi_x^* \xi_x + \xi_\theta^* \xi_\theta + \xi_z^* \xi_z + \frac{c_v \theta^*}{T_0} \theta \right)$$
$$+ \left(\frac{1}{l} \frac{\partial \xi_x^*}{\partial x} + \frac{(1 - \eta)}{\zeta x} \xi_x^* + \frac{\partial \xi_z^*}{\partial z} \right) \tilde{\eta} + \frac{1}{l} \frac{\partial \tilde{\eta}^*}{\partial x} \xi_x$$
$$+ \frac{\partial \tilde{\eta}^*}{\partial z} \xi_z - 2\rho \xi_x \xi_\theta^* \Omega_1 g(x) + \rho \alpha \tilde{\theta}^* \Omega_1^2 g^2$$
$$\times (R_0 + lx)\, \xi_x + 2\rho A \xi_x^* \xi_\theta + \frac{\nu\rho}{2} \Bigg\{ \left(\frac{1}{l} \frac{\partial \xi_x}{\partial x} \right)^2$$
$$+ \left(\frac{1}{l} \frac{\partial \xi_\theta}{\partial x} \right)^2 + 2 \frac{1}{l} \frac{\mathrm{d}V^{(s)}}{\mathrm{d}x} \frac{\partial \xi_\theta}{\partial x} + \left(\frac{1}{l} \frac{\partial \xi_z}{\partial x} \right)^2$$
$$+ \left(\frac{1 - \eta}{l\zeta} \right)^2 \xi_x^2 + \left(\frac{1 - \eta}{l\zeta} \right)^2 \xi_\theta^2 + \frac{2(1 - \eta)}{l\zeta}$$
$$\times \xi_\theta \Omega_1 g + \left(\frac{\partial \xi_\theta}{\partial z} \right)^2 + \left(\frac{\partial \xi_x}{\partial z} \right)^2 + \left(\frac{\partial \xi_z}{\partial z} \right)^2 \Bigg\}$$
$$+ \frac{\rho c_v}{T_0 l} \frac{\mathrm{d}T^{(s)}}{\mathrm{d}x} \xi_x^* \tilde{\theta} + \frac{k}{2T_0} \Bigg\{ \frac{2}{l^2} \frac{\mathrm{d}T^{(s)}}{\mathrm{d}x} \frac{\partial \tilde{\theta}}{\partial x}$$
$$+ \left(\frac{1}{l} \frac{\partial \tilde{\theta}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{\theta}}{\partial z} \right)^2 \Bigg\} \Bigg] (R_0 + lx) l \,\mathrm{d}x\,\mathrm{d}z \quad (50)$$

where terms higher than quadratic in the disturbances and terms containing only non-varied quantities have been neglected.

Let the typical terms in the general fourier expansion for the disturbances be given by:

$$\tilde{\eta}(x, z) = p(x) \cos \frac{az}{l} \quad (51)$$

$$\xi_x(x, z) = u(x) \cos \frac{az}{l} \quad (52)$$

$$\xi_\theta(x, z) = u(x) \cos \frac{az}{l} \quad (53)$$

$$\xi_z(x, z) = -\frac{1}{az} \frac{\mathrm{d}}{\mathrm{d}x} (\zeta u) \sin \frac{az}{l} \quad (54)$$

$$\tilde{\theta}(x, z) = T(x) \cos \frac{az}{l} \quad (55)$$

By choosing the perturbations in this fashion, the terms involving the pressure vanish independent of the choice of p, u, v and T.

Now let the functions u, v and T be approximated by the functions

$$u(x) = \sum_{n=1}^{r} \alpha_n (x^2 - \tfrac{1}{4})^2 x^{(n-1)} \quad (56)$$

$$v(x) = \sum_{n=1}^{r} \beta_n (x^2 - \tfrac{1}{4}) x^{(n-1)} \quad (57)$$

$$T(x) = \sum_{n=1}^{r} \gamma_n (x^2 - \tfrac{1}{4}) x^{(n-1)} \quad (58)$$

These functions were chosen for their completeness and because they satisfy the boundary conditions

$$u = \frac{\mathrm{d}u}{\mathrm{d}x} = v = \tilde{\theta} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2} \quad (59)$$

independent of the arbitrary constants.

The integrations in equation (50) can easily be performed with the aid of equations (51)–(58) and the generalized integrals presented in Appendix I. The resulting equation is a function of the arbitrary constants α_n, α_n^*, β_n, β_n^*, γ_n and γ_n^* and the physical parameters of the problem. The dimensionless parameters which appear in these equations are defined as:

$$Ta = \text{Taylor number} = -\frac{4A_1 \Omega_1^2 l^4}{\nu^2} \quad (60)$$

$$Pr = \text{Modified Prandtl number} = \frac{\rho c_v \nu}{k} \quad (61)$$

Ra = Rayleigh number

$$= -\frac{\alpha Pr (T_2 - T_1) l^4 \Omega_1^2}{\nu^2 \ln(\eta)} \quad (62)$$

N = Ratio of Rayleigh to Taylor number

$$= -\frac{\alpha Pr (T_2 - T_1)}{4A_1 \ln(\eta)} \quad (63)$$

where η and $\tilde{\mu}$ are the radius ratio and the speed ratio, respectively.

The system of equations obtained by differentiation

$$\frac{\partial I_1}{\partial \alpha_n} = 0 \tag{64}$$

$$\frac{\partial I_1}{\partial \beta_n} = 0\,(n = 1, 2, 3, \ldots, r) \tag{65}$$

$$\frac{\partial I_1}{\partial \gamma_n} = 0 \tag{66}$$

with respect to the arbitrary constants must then be made to satisfy the subsidiary conditions that

$$\alpha_n = \alpha^* \tag{67}$$

$$\beta_n = \beta_n^* \tag{68}$$

$$\gamma_n = \gamma_n^* \tag{69}$$

$$\tilde{\sigma} = \tilde{\sigma}^*. \tag{70}$$

For a non-trivial solution to exist the determinant of coefficients in this system of equations must vanish thus yielding a secular equation of the form

$$F(\eta, \tilde{\mu}, \tilde{\sigma}, Ta, N, a) = 0 \tag{71}$$

IV. RESULTS AND CONCLUSIONS

With the aid of an IBM 7040 computer results were obtained using up to three arbitrary constants in each expansion [equations (56–58)] for a wide range of the parameters η, $\tilde{\mu}$, and N. In each case for a given N, η, and $\tilde{\mu}$, the state of marginal stability (i.e. $\tilde{\sigma} = 0$) yields an equation for T as a function of (a). This equation was then solved for the value of (a) which made T a minimum. In all cases only the real roots were considered and in the case of multiple real roots the minimum root was the root of interest.

The results obtained by the variational method for a wide range of the parameters η, $\tilde{\mu}$ and N are presented in Table 1. These results should be compared with the results obtained using Galerkin's method by Diprima, Walowit and Tsao [4] which are found in Table 2. Table 3 lists the critical Taylor numbers and wavenumbers for various assigned values of η and $\tilde{\mu}$.

Table 1. Analytical results using variational method. Critical Taylor numbers, Ta, *and corresponding values of wavenumbers,* a, *for various assigned values of η, $\tilde{\mu}$ and Rayleigh/Taylor number,* N. *(Subscripts denote the number of terms used in approximating series.)*

η	$\tilde{\mu}$	a	N	Ta_3
0·9	0	3·13	+1·0	2250·61
		3·13	+0·5	2720·81
		3·13	0·0	3504·62
		3·13	−0·5	4794·83
		3·13	−1·0	7600·10
0·5	0	3·15	1·0	4615·30
		3·15	0·5	5321·40
		3·15	0·0	6248·73
		3·15	−0·5	7601·40
		3·15	−1·0	9510·62
0·5	0·2	3·15	1·0	2860·57
		3·15	0·5	3400·03
		3·15	0·0	4187·72
		3·15	−0·5	5420·18
		3·15	−1·0	7600·21
0·5	−0·125	3·16	1·0	7172·18
		3·16	0·5	8137·42
		3·16	0·0	9027·32
		3·16	−0·5	9768·51
		3·16	−1·0	10 845·63

In Fig. 1 the variation of Taylor number with the ratio of the Rayleigh to Taylor numbers for various values of η and $\tilde{\mu}$ is shown. This figure exhibits the destabilizing effect of a positive temperature gradient. It also shows that decreasing $\tilde{\mu}$ for fixed η produces a more stable situation.

In Fig. 2 the variation of $Ta\ \eta^2$ with η is shown for various values of $\tilde{\mu}$. For a given value of $\tilde{\mu}$ it becomes evident that increasing the gap size has a destabilizing effect. For a fixed gap size, decreasing $\tilde{\mu}$ has a stabilizing effect.

In Fig. 3 the per cent error based on the results of Chandrasekhar and Elbert [10] is shown vs.

Table 2. Analytical results of Diprima, Walowit and Tsao [4] using Galerkin method. Critical Taylor numbers, Ta, *and corresponding values of wavenumber,* a, *for various assigned values of η, $\tilde{\mu}$ and Rayleigh/Taylor number. (Subscripts denote the number of terms used in approximating series.)*

η	$\tilde{\mu}$	a	N	Ta_3
1·0	0	3·12	+1·0	2177·3
		3·12	+0·5	2653·3
		3·12	0·0	3394·6
		3·12	−0·5	4705·9
		3·12	−1·0	7631·9
0·5	0	3·15	+1·0	4609·0
		3·15	+0·5	5295·3
		3·15	0·0	6218·2
		3·15	—0·5	7522·2
		3·15	−1·0	9494·8
0·5	0·2	3·15	1·0	2852·3
		3·15	0·5	3382·8
		3·15	0·0	4154·2
		3·15	−0·5	5376·8
		3·15	−1·0	7598·0

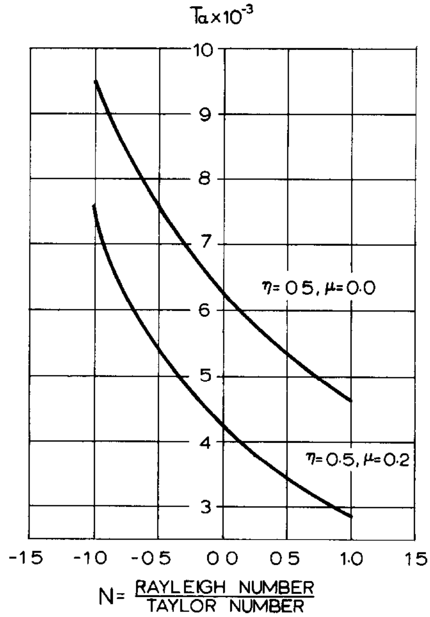

FIG. 1. Variation of Taylor number Ta with ratio of Rayleigh to Taylor number N for assigned values of $\tilde{\mu}$ and η.

Table 3. Critical Taylor numbers and wavenumbers for various assigned values of η and $\tilde{\mu}$. (Subscripts denote the number of terms used in approximating series.)

η	$\tilde{\mu}$	a	$(Ta\eta^2)_3$
0·9	0·20	3·13	2482·8
	0·00	3·13	2838·6
	−0·125	3·14	3447·5
0·8	0·20	3·13	2036·5
	0·00	3·13	2564·3
	−0·125	3·15	3092·8
0·7	0·20	3·13	1673·2
	0·00	3·14	2203·6
	−0·125	3·15	2711·3
0·5	0·20	3·15	1046·7
	0·00	3·15	1562·2
	−0·125	3·18	2233·3

the number of terms in the approximating series for various values of η and $\tilde{\mu}$. In the narrow gap ($1{\cdot}0 < \eta < 0{\cdot}8$) the effect of $\tilde{\mu}$ upon the error was small. However, as the gap becomes wider ($\eta \to 0{\cdot}5$), the effect of $\tilde{\mu}$ becomes larger and more terms become necessary to obtain convergence.

Since the general functional is capable of treating non-linear problems, the significance

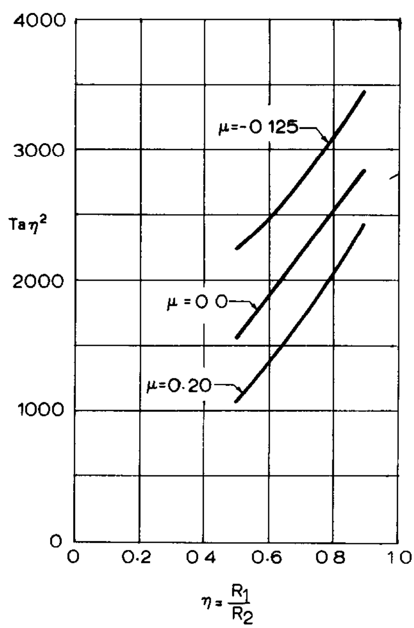

FIG. 2. Variation of $Ta\,\eta^2$ with η when $\tilde{\mu}$ = 0·2, 0·0, −0·125.

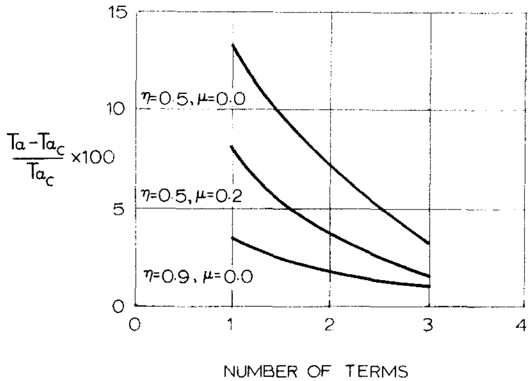

FIG. 3. Per cent error based on exact results of Chandrasekhar and Elbert [10] vs. the number of terms in approximate series for various values of η and $\tilde{\mu}$ when $N = 0$.

of this method is in no way limited by its linearized application here to the Taylor problem. However, it is felt, due to the excellent agreement between the results obtained by the variational method and Galerkin's method, that there is a direct relationship between Galerkin's method and the variational method in the case of linear problems. This confirms the ideas of Roberts [23].

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APPENDIX I

Generalized integrals:

Let

$$S(m, n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^m (x^2 - \tfrac{1}{4})^n \, dx \tag{72}$$

when m = odd

$$= \left\{ \frac{(-1)^n n! \, (\frac{1}{2})^{m+n}}{(m+1)(m+3)\ldots.(m+2n+1)} \right. (m = \text{even}) \tag{73}$$

Let

$$F(m, n, p, \eta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^m (x^2 - \frac{1}{4})^n}{\zeta^p} \, dx \tag{74}$$

when m = even

$$F(m, n, p, \eta) = \left(\frac{1}{\eta_0}\right)^p \sum_{k=0}^{\infty} \left(\frac{\varepsilon}{2\eta_0}\right)^{2k} \frac{(p-1+2k)!}{(p-1)!(2k)!} \times S(m+2k, n) \tag{75}$$

when m = odd

$$F(m, n, p, \eta) = -\left(\frac{1}{\eta_0}\right)^p \sum_{k=0}^{\infty} \left(\frac{\varepsilon}{2\eta_0}\right)^{(2k+1)} \frac{(p+2k)!}{(p-1)!(2k+1)!} \times S(m+2k+1, n) \tag{76}$$

where

$$\eta_0 = \frac{1+\eta}{2} \tag{77}$$

$$\varepsilon = 1 - \eta \tag{78}$$

$$\zeta = \frac{(1+\eta)}{2} + (1-\eta)x$$

$$= \varepsilon x + \eta_0 \tag{79}$$

SOLUTION VARIATIONNELLE DU PROBLÈME DE STABILITÉ DE TAYLOR BASÉE SUR LA THERMODYNAMIQUE EN NON-ÉQUILIBRE

Résumé—Les articles récents de Glansdorff, Prigogine et Schechter ont montré qu'il existe un potentiel local dépendant du temps dans lequel l'évolution temporelle macroscopique correspond à un extremum. Schechter et Himmelblau [19] ont résolu le problème bien connu de Bénard en employant une formulation variationnelle restreinte. Dans cet article, les travaux de Glansdorff, Prigogine et Schechter sont modifiés et étendus pour obtenir une solution variationnelle au problème de stabilité de Taylor pour un fluide visqueux contenu entre des cylindres coaxiaux en rotation avec et sans gradient radial de température. Les résultats obtenus sont en excellent accord avec les résultats expérimentaux de Chandrasekhar [9], Chandrasekhar et Elbert [10], Diprima, Walowit et Tsao [4], Yih [5], Lai [25], et Becker et Kaye [7] et avec les résultats expérimentaux de Becker et Kaye [8], Bjorklund et Kaye [12], Haas et Nissan [13] et Ho, Nardacci et Nissan [15].

EINE VARIATIONSLÖSUNG ZUM PROBLEM DER TAYLOR-STABILITÄT AUF GRUND DER NICHT-GLEICHGEWICHTSTHERMODYNAMIK

Zusammenfassung—Die jüngsten Arbeiten von Glansdorff, Prigogine und Schechter haben gezeigt, dass ein zeitabhängiges örtliches Potential existiert, in welchem die makroskopische Zeitenwicklung einem Extremum entspricht. Schechter und Himmelblau [19] lösten das wohlbekannte Benard-Problem unter Verwendung einer eingeschränkten Variationsformulierung. In dieser Untersuchung weiden die Arbeiten von Glansdorff, Prigogine und Schechter modifiziert und so erweitert, dass man hinsichtlich des Taylor–Stabilitätsproblems für eine, zwischen rotierenden koaxialen Zylindern befindliche zähe Flüssigkeit mit und ohne radialen Temperaturgradienten Variationslösungen erhält. Die erhaltenen Ergebnisse stimmen ausgezeichnet mit den analytischen Resultaten von Chandrasekhar [9], Chandrasekhar und Elbert [10], Diprima, Walowit, Tsao [4], Yih [5], Lai [25], Beeker und Kaye [7] und mit den Messergebnissen von Becker und Kaye [8], Bjorklund und Kays [12], Haas und Nissan [13], Ho, Nardacci und Nissan [15] überein.

ВАРИАЦИОННОЕ РЕШЕНИЕ СТАЦИОНАРНОЙ ЗАДАЧИ ТЕЙЛОРА НА ОСНОВЕ НЕРАВНОВЕСНОЙ ТЕРМОДИНАМИКИ

Аннотация—В недавно опубликованных работах Гландорфа, Пригожина и Шехтера показано, что существует зависящий от времени локальный потенциал, в котором макроскопическое время развития имеет экстремум. Шехтер и Химмелблау (19) решали хорошо известную задачу Бенарда, используя вариационную задачу в ограниченной постановке. В этой статье данные Гландсдорфа, Пригожина и Шехтера модифицированы и расширены для того, чтобы получить решение вариационной задачи Тэйлора по устойчивости вязкой жилдкости, находящейся между вращающимися коаксиальными цилиндрами при наличии и отсутствии радиального градиента температуры.